The effect of subfilter-scale properties on regularization models

Lagrangian-averaged modeling for Navier-Stokes & MHD

J. Pietarila Graham,[1] Darryl Holm,[2,3] Pablo Mininni,[4,5] and Annick Pouquet[4]

[1]Max-Planck-Institut für Sonnensystemforschung
[2]Department of Mathematics, Imperial College London
[3]Computer and Computational Science Division, Los Alamos National Laboratory
[4]National Center for Atmospheric Research
[5]Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires

QLES2009

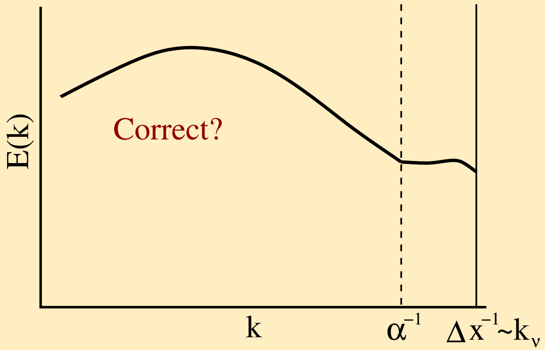
What is a regularization SGS model?

Definition: regularization model

- Modification of nonlinear (not dissipative) terms
- Unique, smooth/regular solutions $\forall t$ even for lim $\nu \to 0$
- Original fluid equations lim $\alpha \equiv \text{filter width} \to 0$

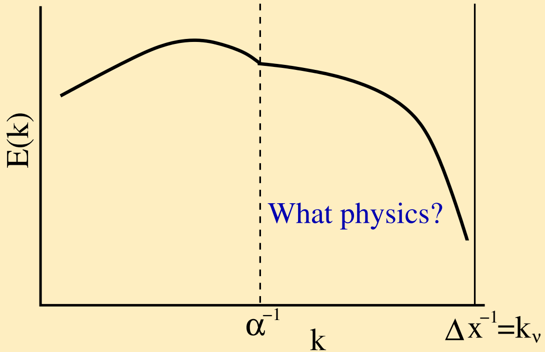
1 - Do the models work?

Do sub-filter-scale physics reproduce super-filter-scale properties?



2 - **HOW** do the models work?

What are the sub-filter-scale physics?



Lagrangian-averaged Navier-Stokes (LANS, $\alpha-$model)

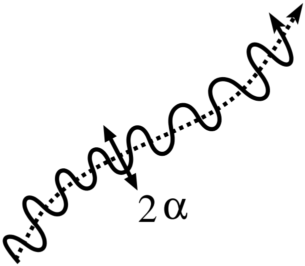
Camassa et al. 1993, Holm et al. 1998, Chen et al. 1998

What is the model?

1. Generalized Lagrangian mean (Andrews & McIntyre 1978)
2. Taylor's frozen-in-turbulence

Mathematically

- Retains Hamiltonian structure
- Preserves Kelvin's theorem, small-scale circulation
- Conservation of energy, helicity (H^1_α *not* L^2: $\frac{1}{2}\langle \bar{\mathbf{v}} \cdot \mathbf{v} \rangle$ *not* $\frac{1}{2}\langle v^2 \rangle$)



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What is the model?

1. Generalized Lagrangian mean (Andrews & McIntyre 1978)
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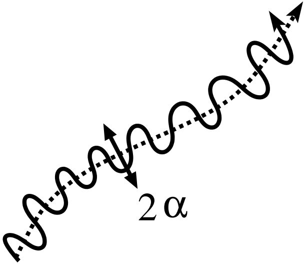
Physically

- Retains non-local large-small interactions
- Limits small local interactions
- Reduces flux of energy in sub$-\alpha$ scales



Lagrangian-averaged Navier-Stokes (LANS, $\alpha-$model)

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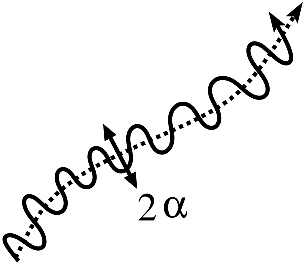
Equations

$$\partial_t v_i + \partial_j(\bar{v}_j v_i) + \partial_i \pi + v_j \partial_i \bar{v}_j = \nu \partial_{jj} v_i$$
$$\partial_j v_j = \partial_j \bar{v}_j = 0$$
Filter: $v_i = (1 - \alpha^2 \partial_{jj}) \bar{v}_i$

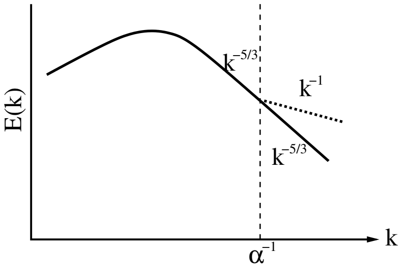
LES form

$$\partial_t \bar{v}_i + \partial_j(\bar{v}_j \bar{v}_i) + \partial_i \bar{P} + \partial_j \bar{\tau}_{ij}^{\alpha} = \nu \partial_{jj} \bar{v}_i$$
SGS:
$$\bar{\tau}_{ij}^{\alpha} = (1 - \alpha^2 \partial_{jj})^{-1} \alpha^2 (\partial_m \bar{v}_i \partial_m \bar{v}_j + \partial_m \bar{v}_i \partial_j \bar{v}_m - \partial_i \bar{v}_m \partial_j \bar{v}_m)$$



LANS α – *model*: How does it work?

$H^1_\alpha \sim k^{-1}$ (Holm 2002)



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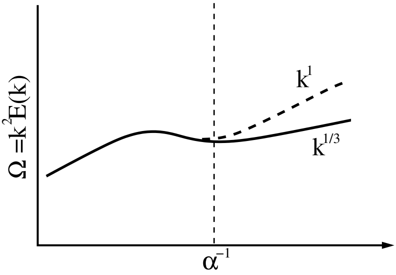

Dissipates faster in k

$-\frac{dE}{dt} = \varepsilon = 2\nu\Omega \sim \frac{1}{Re} \int^{k_\nu} k^2 E(k) dk$
$E(k)dk \sim \varepsilon^\gamma k^\beta$
$k_\nu \sim Re^{1/(3+\beta)} \quad \beta = -5/3 \quad$ or $\; -1$
$dof_\alpha \sim \alpha^{-1} Re^{3/2}$
(predicted Foias et. al 2001, confirmed Graham et al. 2007)
$dof_{NS} \sim Re^{9/4}$

LANS α – *model*: At what *Re*?

Great at moderate *Re*

- Better than dynamic eddy viscosity ($Re_\lambda \approx 220$, Mohseni et al. 2003)
- Better than dynamic mixed (similarity) eddy viscosity ($Re \approx 50$, Geurts & Holm 2006)

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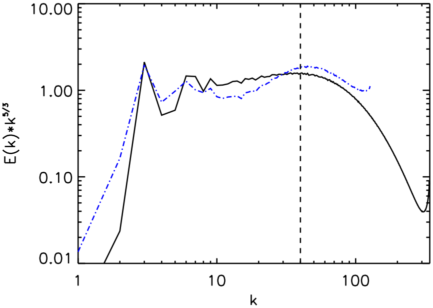
Forced TG $k = 2$, $Re \approx 3300$

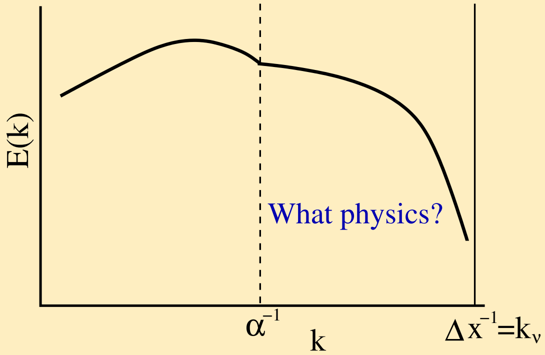


Navier-Stokes 1024^3

LANS 384^3, $\alpha = 2\pi/40$

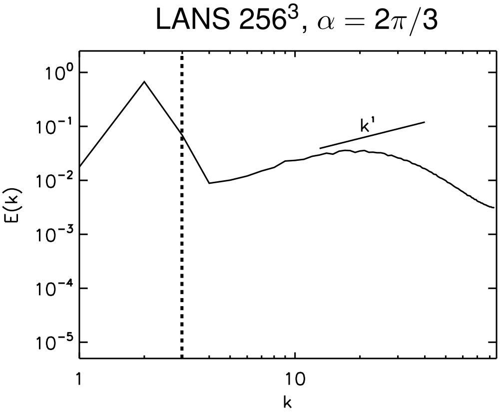
2 - **HOW** do the models work?

What are the sub-filter-scale physics?



LANS α − *model*: How does it fail?

Graham et al. PRE **76**, 056310 (2007)



Forced TG $k = 2$, *Re* ≈ 8000

Rigid bodies

$\delta\bar{\mathbf{v}}(\mathbf{l}) = \mathbf{\Omega} \times \mathbf{l}$

$\delta\bar{v}_{||}(l) = \delta\bar{\mathbf{v}}(\mathbf{l})\cdot\mathbf{l}/l = 0$

$\langle(\delta\bar{v}_{||})^3\rangle = 0$

$\delta\bar{v}^2 \sim l^0$

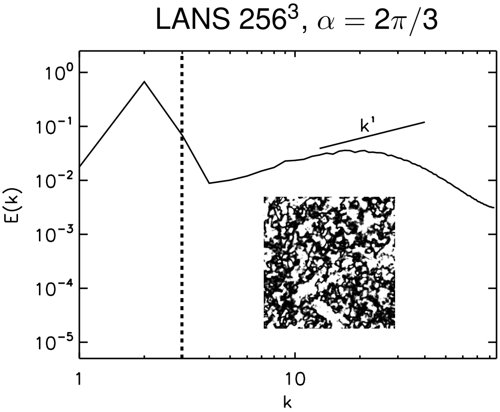
$\bar{v} \sim \alpha^{-2}k^{-2}v$

$E_\alpha(k)k \sim \bar{v}v \sim k^2$

$E_\alpha(k) \sim k^1$

LANS α − *model*: How does it fail?

Graham et al. PRE **76**, 056310 (2007)



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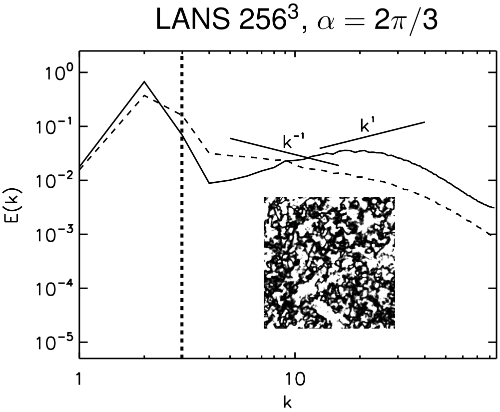

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How to get rid of rigid bodies?

Change regularization

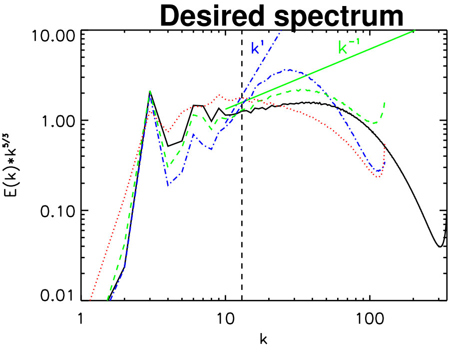
- Truncate LANS$-\alpha$
  $\bar{\tau}^{\alpha}_{ij} = (1 - \alpha^2 \partial_{jj})^{-1} \alpha^2 (\partial_m \bar{v}_i \partial_m \bar{v}_j + \partial_m \bar{v}_i \partial_j \bar{v}_m - \partial_i \bar{v}_m \partial_j \bar{v}_m)$
- 1 term Clark$-\alpha$ (Cao et al. 2005)
- 2 terms Leray$-\alpha$ (Geurts & Holm 2002, 2003, 2006; Cheskidov et al. 2005)
- Conserves H^1_α, L^2 energy but *not* helicity, circulation

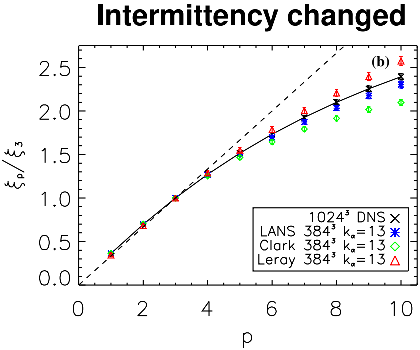
Clark$-\alpha$, Leray$-\alpha$: Sub-filter-scale properties

Graham et al. Phys. Fluids **20**, 035107 (2008)

Navier-Stokes 1024^3

Clark$-\alpha$, Leray$-\alpha$, LANS 384^3, $\alpha = 2\pi/13$





Forced TG $k = 2$, $Re \approx 3300$, $Re_\lambda \approx 790$

What about MHD?

Circumvents rigid body formation?

- Source term in Kelvin's circulation theorem
  $\frac{d}{dt}\Gamma = \frac{d}{dt}\oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{j} \times \mathbf{b} \cdot d\mathbf{r}$
- Spectrally **nonlocal** interactions between large scale of one field and small scale of the other (Alexakis et al. 2005; Alexakis 2007)

LAMHD$-\alpha$ (MHD$-\alpha$)

Holm 2002, Montgomery & Pouquet 2002

Equations

$$\partial_t \mathbf{v} + \boldsymbol{\omega} \times \bar{\mathbf{v}} = \mathbf{j} \times \bar{\mathbf{b}} - \boldsymbol{\nabla}\pi + \nu\nabla^2\mathbf{v}$$
$$\partial_t \bar{\mathbf{b}} = \boldsymbol{\nabla} \times (\bar{\mathbf{v}} \times \bar{\mathbf{b}}) + \eta\nabla^2\mathbf{b}$$
$$\boldsymbol{\nabla} \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot \bar{\mathbf{v}} = \boldsymbol{\nabla} \cdot \mathbf{b} = \boldsymbol{\nabla} \cdot \bar{\mathbf{b}} = 0$$

Filter: $\mathbf{v} = (1 - \alpha^2\nabla^2)\bar{\mathbf{v}}$, $\mathbf{b} = (1 - \alpha^2\nabla^2)\bar{\mathbf{b}}$

Properties

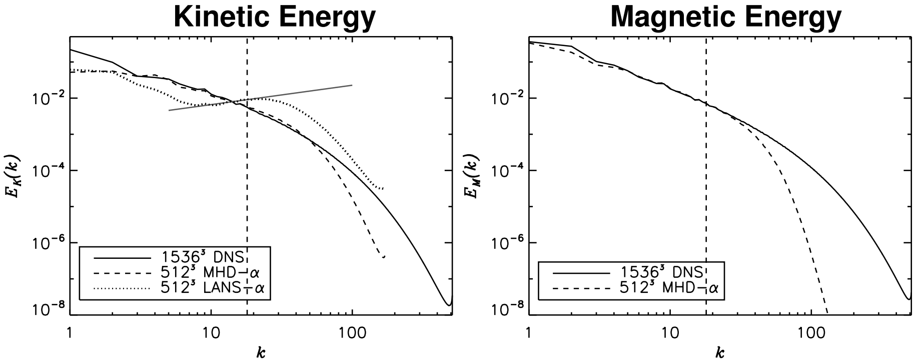
- Math
  - Preserves ideal MHD invariants (H^1_α *not* L^2)
  - Alfvén's theorem
- Physics
  - Supports Alfvén waves at all scales
  - Wavelengths $< \alpha$: slows & damps

LAMHD$-\alpha$: No positive power laws; No contamination

Graham et al. PRE **80**, 016313 (2009)

MHD 1536^3

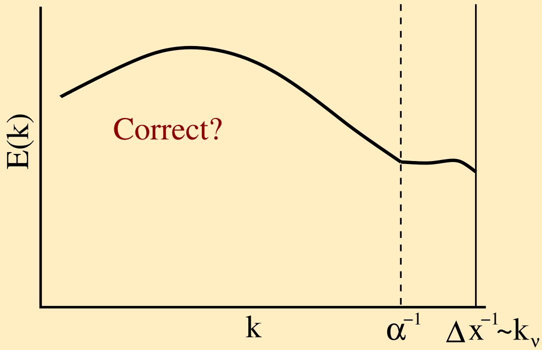
LANS, LAMHD 512^3, $\alpha = 2\pi/18$



Decay ABC $k \in [1, 4]$ + *noise*, $Re \approx 9200$, $Re_\lambda \approx 1100$

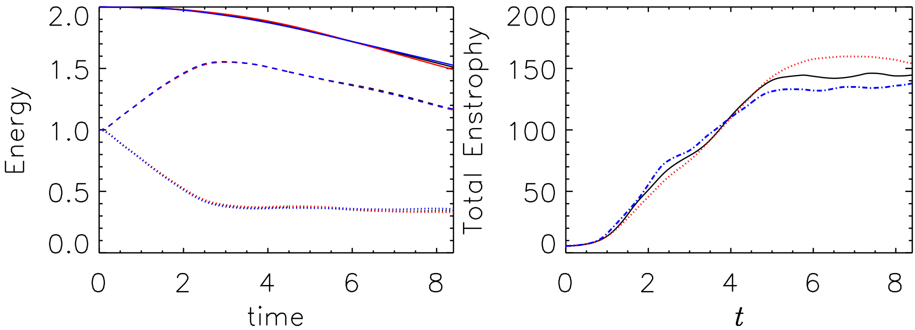
1 - Do the models work?

Do sub-filter-scale physics reproduce super-filter-scale properties?



MHD$-\alpha$ SGS test: Global quantities

DNS 1024^3
MHD 168^3, LAMHD 168^3 $\alpha = 2\pi/28$

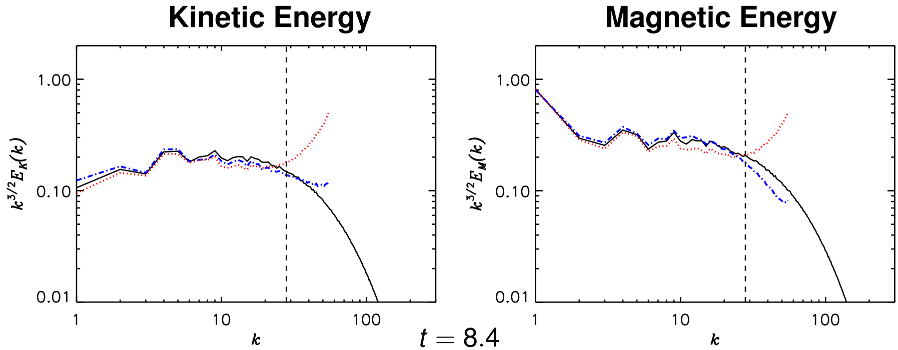


Decay ABC $k \in [1, 4]$, $Re \approx 3300$

MHD$-\alpha$ SGS test: Better spectra

DNS 1024^3

MHD 168^3, LAMHD 168^3 $\alpha = 2\pi/28$

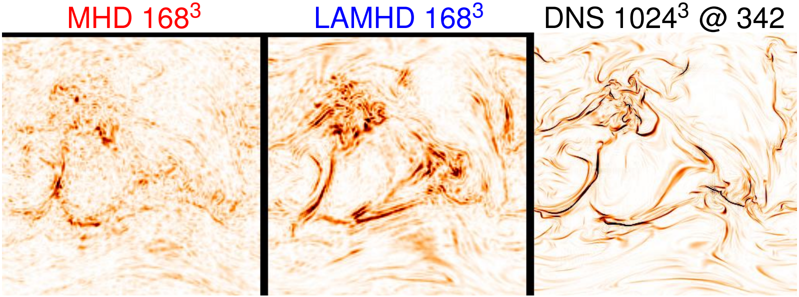


$t = 8.4$

Decay ABC $k \in [1, 4]$, $Re \approx 3300$

MHD$-\alpha$ SGS test: Captures current sheets

Square current, j^2



$t = 8.4$

Conclusions

Lagrangian-averaged Navier-Stokes α

- Conserves small-scale circulation
- Prohibits local small-scale to small-scale interactions
- Develops rigid bodies $\rightarrow$ spectral contamination

Lagrangian-averaged Magnetohydrodynamics α

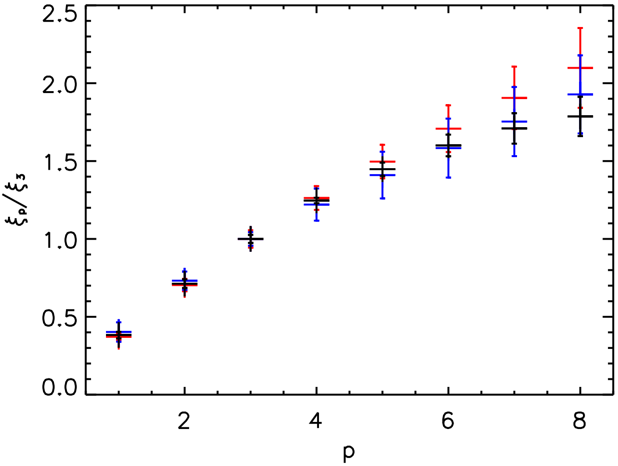
- Lorentz force is source of circulation and conduit for *nonlocal* interactions
- Only damps small-wavelength Alfvén waves & local small-scale interactions
- May be viable SGS

Previous tests

2D[†]	time evolution of energies	✓
	time evolution of cross-helicity	$\approx$
	energy spectra	+
	dynamic alignment	$\approx$
	PDFs	except tails
	inverse cascade of vector potential	$<$
3D[‡]	time evolution of energies	✓
	time evolution of magnetic helicity	$\approx$
	energy spectra	✓
	dynamic alignment	$<$
	inverse cascade of magnetic helicity	$<$
	dynamo	✓

[†] Mininni et al. *Phys. Fluids* **17**, 035112 (2005). [‡] Mininni et al. *Phys. Rev. E* **71**, 046304 (2005), Ponty et al. *Phys. Rev. Lett.* **94**, 164502

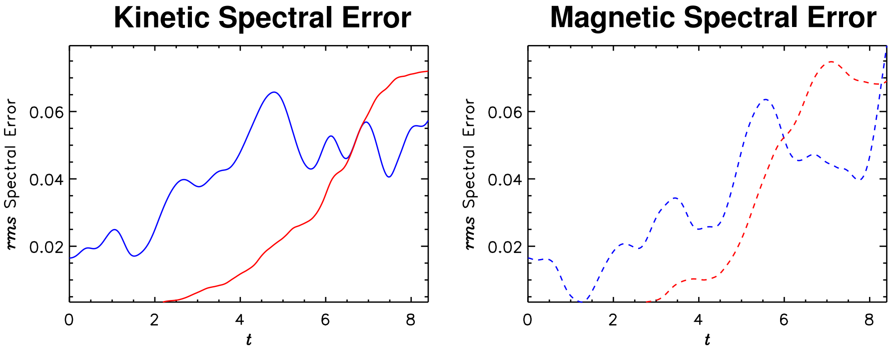
MHD$-\alpha$ SGS test: Better intermittency



MHD$-\alpha$ SGS test: Better spectra

MHD 168^3, LAMHD 168^3 $\alpha = 2\pi/28$

ϵ_0^b, Meyers et al. 2006



No general LES for MHD

Challenges

- Eddy-viscosity $\leftrightarrow k^{-5/3}$ (Chollet & Lesieur 1981) *not -3/2*
- E_K & E_M *not* conserved quantities
- Spectrally **nonlocal** interactions between large scale of one field and small scale of the other (Alexakis et al. 2005; Alexakis 2007)
- Unresolved **v** & **b** interactions
- Many regimes – no generally applicable MHD-LES

No general LES for MHD

Existing Models

- Dissipative LES (Theobald et al 1994)
  - Ignore sub-filter scale energy exchanges
  - Assumes energy spectra of non-conserved quantities
- Dissipative LES (Zhou et al 2002)
  - non-helical, stationary MHD
  - $k^{-5/3}$ and fixed ratio of energies
- Cross-helicity model (Müller & Carati 2002)
  - Assumes alignment between the fields
  - Reduced intermittency
- Low Re_M LES (Ponty et al 2004)
- Hyper-resistivity (not LES - Haugen & Brandenburg 2006)
  - Requires recalibration of length scales to known DNS

LAMHD$-\alpha$: No rigid bodies

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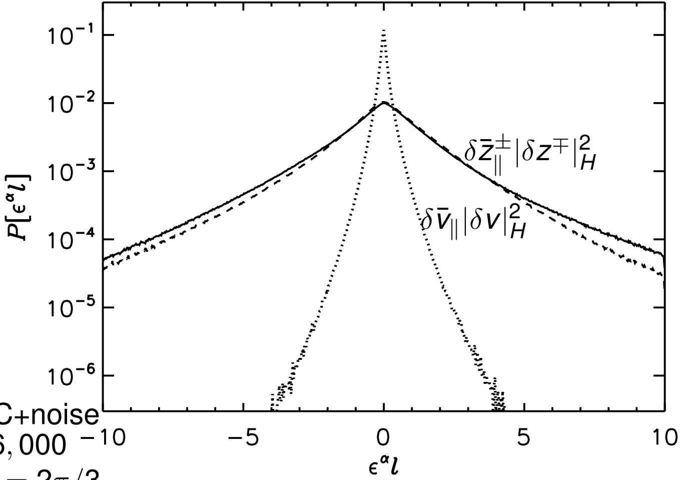
PDF of flux to small scales



Decay ABC+noise
$Re \approx 26,000$
256^3, $\alpha = 2\pi/3$